A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics and Statistics Part C: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

Trinity Term 2016

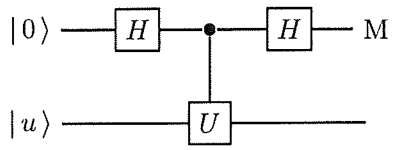
FRIDAY, 10 JUNE 2016, 9.30am to 11.00am

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. Consider the following quantum network composed of the two Hadamard gates, one controlled-U operation and the measurement M in the computational basis,



The top horizontal line represents a qubit and the bottom one an auxiliary physical system.

(a) [6 marks] Suppose $|u\rangle$ is an eigenvector of U, such that $U|u\rangle = e^{i\alpha}|u\rangle$. Step through the execution of this network, writing down quantum states of the qubit and the auxiliary system after each computational step. What is the probability for the qubit to be found in state $|0\rangle$?

Regardless the state of the auxiliary system, the probability P_0 for the qubit to be found in state $|0\rangle$, when the measurement M is performed, can be written as

$$P_0 = \frac{1}{2}(1 + v\cos\phi),$$

where v and ϕ depend on U and on the initial state of the auxiliary system.

(b) [9 marks] Show that for an arbitrary pure state $|u\rangle$ of the auxiliary system the quantities v and ϕ are given by the relation $ve^{i\phi} = \langle u|U|u\rangle$.

(c) [7 marks] Suppose the auxiliary system is prepared in a mixed state described by the density operator ρ,

$$\rho = p_1|u_1\rangle\langle u_1| + p_2|u_2\rangle\langle u_2| + \ldots\ldots + p_n|u_n\rangle\langle u_n|,$$

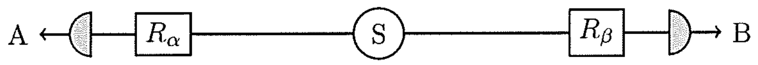
where vectors $|u_k\rangle$ form an orthonormal basis, $p_k \geqslant 0$ and $\sum_{k=1}^{n} p_k = 1$. Show that

$$ve^{i\phi} = \mathrm{Tr}(\rho U).$$

(d) [3 marks] How would you modify the network in order to estimate $\mathrm{Tr}(\rho U)$? How would you estimate $\mathrm{Tr}\,U$?

2. (a) [4 marks] Two entangled qubits in state $\frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle)$ are generated by source S; one qubit is sent to Alice and one to Bob, who perform measurements in the computational basis. What is the probability that Alice and Bob will register identical results? Can any correlations they observe be used for instantaneous communication?

(b) [8 marks] Prior to the measurements in the computational basis Alice and Bob apply unitary operations R_α and R_β to their respective qubits



The gate R_θ is defined by its action on the basis states

$$
\begin{aligned}
|\,0\,\rangle &\rightarrow \cos\theta|\,0\,\rangle + \sin\theta|\,1\,\rangle, \\
|\,1\,\rangle &\rightarrow -\sin\theta|\,0\,\rangle + \cos\theta|\,1\,\rangle.
\end{aligned}
$$

Show that the state of the two qubits prior to the measurements is

$$
\cos(\alpha-\beta)\frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle) - \sin(\alpha-\beta)\frac{1}{\sqrt{2}}(|\,01\,\rangle - |\,10\,\rangle)
$$

What is the probability that Alice and Bob's outcomes are identical?

(c) [5 marks] Let A_1, A_2, B_1 and B_2 be the measurements defined by the settings $\alpha_1 = 0$, $\alpha_2 = \frac{2\pi}{8}$, $\beta_1 = \frac{\pi}{8}$ and $\beta_2 = \frac{3\pi}{8}$, respectively. Alice and Bob perform a statistical test in which Alice repeatedly measures either A_1 or A_2 and Bob either B_1 or B_2. For each run they choose their settings randomly and independently from each other and check whether one of the following condition is satisfied

$$
A_1 = B_1, \quad B_1 = A_2, \quad A_2 = B_2, \quad B_2 \neq A_1.
$$

What is the probability (the fraction of the measurements) in which they find the outcomes in agreement with the four conditions?

(d) [4 marks] An adversary, Eve, who controls the source S, claims she has a way to predict outcomes A_1, A_2, B_1 and B_2 with certainty. Can such outcomes have pre-determined values?

(e) [4 marks] Explain how Alice and Bob, miles apart but able to communicate over a public channel, can use the scheme to establish a secret cryptographic key.

3. The controlled-not is a two-qubit gate defined in the computational basis as

$$|x\rangle|y\rangle \mapsto |x\rangle|x \oplus y\rangle,$$

where $x, y = 0, 1$.

(a) [4 marks] If the second qubit is prepared in state $|0\rangle$ ($y = 0$) the gate clones the bit value of the first qubit $|x\rangle|0\rangle \mapsto |x\rangle|x\rangle$. Show that this does not imply that $|\psi\rangle|0\rangle \mapsto |\psi\rangle|\psi\rangle$ for any quantum state $|\psi\rangle$ of the first qubit.

(b) [6 marks] A universal quantum cloner is a hypothetical quantum device that operates on two qubits and on some auxiliary system. Given one qubit in any quantum state $|\psi\rangle$ and the other one in a prescribed state $|0\rangle$ it maps

$$|\psi\rangle|0\rangle|R\rangle \mapsto |\psi\rangle|\psi\rangle|R'\rangle,$$

where $|R\rangle$ and $|R'\rangle$ are, respectively, the initial and the final state of any other auxiliary system that may participate in the cloning process ($|R'\rangle$ may depend on $|\psi\rangle$). Show that such a cloner is impossible.

(c) [3 marks] The best approximation to the universal quantum cloner is the following transformation

$$|\psi\rangle|0\rangle|0\rangle \mapsto \sqrt{\frac{2}{3}}|\psi\rangle|\psi\rangle|\psi\rangle + \sqrt{\frac{1}{6}}\left(|\psi\rangle|\psi^{\perp}\rangle + |\psi^{\perp}\rangle|\psi\rangle\right)|\psi^{\perp}\rangle$$

where $|\psi^{\perp}\rangle$ is a normalised state vector orthogonal to $|\psi\rangle$ and the auxiliary system is another qubit. Explain why the reduced density matrices of the first and the second qubit must be identical.

(d) [9 marks] Show that the reduced density matrix of the first (and the second) qubit can be written as

$$\rho = \frac{5}{6}|\psi\rangle\langle\psi| + \frac{1}{6}|\psi^{\perp}\rangle\langle\psi^{\perp}|.$$

(e) [3 marks] What is the probability that the clone in state ρ will pass a test for being in the original state $|\psi\rangle$?