A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

Trinity Term 2017

WEDNESDAY, 7 JUNE 2017, 9.30am to 11.15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [2 marks] Describe the *Hadamard gate* and the *controlled*-NOT *gate.*

   (b) [3 marks] Draw a quantum network (circuit) that encodes a single qubit state $\alpha|0\rangle+\beta|1\rangle$ into the state $\alpha|00\rangle+\beta|11\rangle$ of two qubits, where $\alpha, \beta \in \mathbb{C}$.

   (c) [5 marks] Two qubits were prepared in state $\alpha|00\rangle+\beta|11\rangle$, exposed to bit flip-errors, and then measured with an ancillary qubit, as shown in Fig. 1. The result of the measurement is x. Can you infer the absence of errors when $x=0$? Can you infer the presence of errors when $x=1$? Can you correct any detected errors?

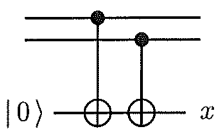


Fig. 1

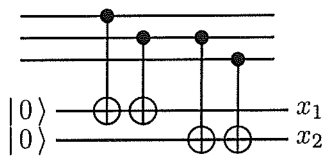


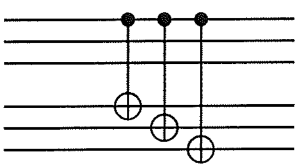
Fig. 2

Three qubits were prepared in state $\alpha|000\rangle+\beta|111\rangle$ and then, by mistake, someone applied the Hadamard gate to one of them, but nobody remembers which one. Your task is to recover the original state of the three qubits.

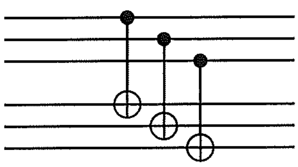
   (d) [4 marks] Express the Hadamard gate as the sum of two Pauli matrices. Pick up one of the three qubits and apply the Hadamard gate. How is the state $\alpha|000\rangle+\beta|111\rangle$ modified? Interpret this in terms of bit-flip and phase-flip errors.

   (e) [6 marks] You perform the error syndrome measurement shown in Fig. 2. Suppose the outcome of the measurement is $x_1=0, x_2=1$. How would you recover the original state? Describe the recovery procedure when $x_1=0, x_2=0$.

The figure below shows two implementations of a controlled-NOT gate acting on the encoded states of the three qubit code.



Implementation A



Implementation B

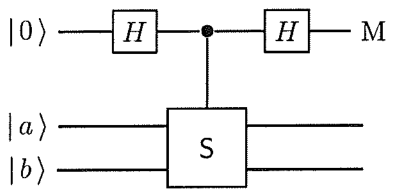
   (f) [5 marks] Assume that the only sources of errors are individual controlled-NOT gates which produce bit-flip errors in their outputs. These errors are independent and occur with a small probability p. For each of the two implementations find the probability of generating unrecoverable errors at the output. Which of the two implementations is fault-tolerant?

2. The swap gate S on two qubits is defined first on product vectors, $S : |\,a\,\rangle|\,b\,\rangle \mapsto |\,b\,\rangle|\,a\,\rangle$ and then extended to sums of product vectors by linearity.

   (a) [3 marks] Show that the four Bell states $\frac{1}{\sqrt{2}}(|\,00\,\rangle \pm |\,11\,\rangle)$, $\frac{1}{\sqrt{2}}(|\,01\,\rangle \pm |\,10\,\rangle)$ are eigenvectors of S which form the orthonormal basis in the Hilbert space associated with two qubits. Which Bell states span the symmetric subspace (all eigenvectors of S with eigenvalue 1) and which the antisymmetric one (all eigenvectors of S with eigenvalue -1)? Can S have any other eigenvalues except ± 1?

   (b) [3 marks] Show that $P_{\pm} = \frac{1}{2}(\mathbb{1} \pm S)$ are two orthogonal projectors which form the decomposition of the identity and project on the symmetric and the antisymmetric subspaces. Decompose the state vector $|\,a\,\rangle|\,b\,\rangle$ of two qubits into symmetric and antisymmetric components.

   (c) [7 marks] Consider the following quantum network composed of the two Hadamard gates, one controlled-S operation (also known as the controlled-swap or the Fredkin gate) and the measurement M in the computational basis,



   The state vectors $|\,a\,\rangle$ and $|\,b\,\rangle$ are normalised but not orthogonal to each other. Step through the execution of this network, writing down quantum states of the three qubits after each computational step. What are the probabilities of observing 0 or 1 when the measurement M is performed?

   (d) [4 marks] Explain why this quantum network implements projections on the symmetric and the antisymmetric subspaces of the two qubits.

   (e) [4 marks] Two qubits are transmitted through a quantum channel which applies the same, randomly chosen, unitary operation U to each of them. Show that $U \otimes U$ leaves the symmetric and antisymmetric subspaces invariant.

   (f) [4 marks] Polarised photons are transmitted through an optical fibre. Due to the variation of the refractive index along the fibre the polarisation of each photon is rotated by the same unknown angle. This makes communication based on polarisation encoding unreliable. However, if you can prepare any polarisation state of two photons you can still use the channel and communicate without any errors. How can this be achieved?

3. Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(a) [3 marks] Check that such a parametrised ϱ has all the mathematical properties of a density matrix. Find the eigenvalues of ϱ and explain why the length of the Bloch vector cannot exceed 1.

(b) [5 marks] Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger$$

where matrices A_k satisfy $\sum_k A_k^\dagger A_k = \mathbb{1}$. Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.

(c) [9 marks] A qubit in state ϱ is transmitted through a depolarising channel that effects a completely positive map

$$\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x\varrho\sigma_x + \sigma_y\varrho\sigma_y + \sigma_z\varrho\sigma_z\right),$$

for some $0 \leqslant p \leqslant 1$. Show that under this map the Bloch vector associated with ϱ shrinks by the factor $(3-4p)/3$.

Consider a map $\mathcal{N}$, called universal-NOT, which acts on a single qubit and inverts its Bloch sphere,

$$\mathcal{N}(\mathbb{1}) = \mathbb{1} \quad \mathcal{N}(\sigma_x) = -\sigma_x \quad \mathcal{N}(\sigma_y) = -\sigma_y \quad \mathcal{N}(\sigma_z) = -\sigma_z$$

(d) [2 marks] Explain why $\mathcal{N}$, acting on a single qubit, maps density matrices to density matrices.

(e) [6 marks] The joint state of two qubits is described by the density matrix

$$\rho = \tfrac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\right).$$

Apply $\mathcal{N}$ to the first qubit leaving the second qubit intact. Write the resulting matrix and explain why $\mathcal{N}$ is not a completely-positive map.

[*You may use without proof that the Pauli matrices σ_x, σ_y, and σ_z are*

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and that they anticommute and square to the identity $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$.]