A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics and Computer Science Part C: Paper C7.4
Honour School of Mathematics and Statistics Part C: Paper C7.4
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4
Honour School of Physics Part C: Paper C7.4
Master of Science in Mathematical Sciences: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4

Introduction to Quantum Information

TRINITY TERM 2023

Friday 02 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

Throughout the entire exam, we refer to the Pauli matrices as $X \equiv \sigma_x$, $Y \equiv \sigma_y$ and $Z \equiv \sigma_z$, where

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They anticommute and square to the identity, $X^2 = Y^2 = Z^2 = \mathbb{1}$.

1. (a) [4 marks] What are the matrix representations of the Hadamard and the controlled-NOT gates in the computational basis? Explain why the Hadamard gates are often used as the first operation in quantum algorithms.

Quantum computation may be viewed as an interference experiment involving many qubits, with phase shifts induced by evaluations of Boolean functions $f : \{0,1\}^n \to \{0,1\}$. These evaluations are usually implemented as unitary evolutions of two registers,

$$|x\rangle\,|y\rangle \to |x\rangle\,|y \oplus f(x)\rangle\,,$$

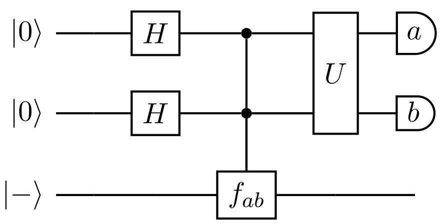
where $x \in \{0,1\}^n$ and $y \in \{0,1\}$.

(b) [3 marks] What is the effect of the quantum function evaluation on the first register if, instead of $|y\rangle$, the second register is prepared in the state $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$?

Let f_{ab} be a Boolean function, $f_{ab} : \{0,1\}^2 \to \{0,1\}$, which takes the value 0 for all its arguments except one binary string $ab \in \{0,1\}^2$, for which it takes the value 1. There are four such functions and one of them is presented to you as an oracle. Your task is to identify the function (to find the binary string ab) with the minimal number of calls to the oracle.

(c) [2 marks] In the classical scenario, how many calls to the oracle are required in order to identify the function?

A quantum algorithm solving this problem with only one call to the oracle is implemented by the following circuit,



Let $|f_{ab}\rangle$ be the state of the first register (the two top qubits) prior to the unitary operation U.

(d) [4 marks] Write down the four possible states $|f_{ab}\rangle$ and explain why they are perfectly distinguishable.

(e) [2 marks] The final unitary operation U maps the four states $|f_{ab}\rangle$ to the computational basis states. Write down the matrix representation of U in the computational basis.

(f) [6 marks] Draw a circuit diagram implementing U. It contains two Hadamard gates and one control-NOT gate, and it ends with two NOT gates (one on each qubit).

(g) [4 marks] Suppose g_{ab} is a Boolean function $g : \{0,1\}^2 \to \{0,1\}$ which takes the value 1 for all its arguments except one binary string ab, for which it takes the value 0. Given an oracle evaluating one of the four functions g_{ab} how would you identify which one it is with only one call to the oracle?

2. A quantum channel acting on the density operator ρ can always be written in the Kraus form,

$$\rho \mapsto \rho' = \sum_{i=1}^{n} K_i \rho K_i^\dagger,$$

where $K_1, \ldots, K_n$ are the Kraus operators, which satisfy $\sum_{i=1}^n K_i^\dagger K_i = \mathbb{1}$.

(a) [4 marks] Show that any quantum channel preserves positivity and trace. Show that a composition of two quantum channels with Kraus operators A_i and B_j is another quantum channel with the Kraus operators $B_j A_i$.

(b) [4 marks] Is any map which preserves positivity and trace a quantum channel?

(c) [4 marks] Given an orthonormal basis $\{|e_i\rangle\}$ show that the projectors on the basis states, $P_i = |e_i\rangle\langle e_i|$, form a set of Kraus operators. Describe the action of the associated quantum channel,

$$\rho \mapsto \rho' = \sum_{i=1}^{n} P_i \rho P_i^\dagger,$$

on any density matrix ρ written in the $\{|e_i\rangle\}$ basis. What are the diagonal and off-diagonal elements of the resulting density matrix ρ'?

Each term in the Kraus decomposition may be interpreted as the result of a measurement. Such a generalised measurement performed on a quantum system in state ρ gives outcome m with probability $p_m = \mathrm{Tr}\left(K_m \rho K_m^\dagger\right)$ and leaves the system in the post-measurement state,

$$\rho \to \frac{1}{p_m} K_m \rho K_m^\dagger.$$

(d) [4 marks] Suppose you performed a generalised measurement, defined by a set of Kraus operators K_i, on a quantum system in state ρ. Your measuring apparatus registered the outcome but you did not look at it, and soon after the record of the outcome was irretrievably lost. From your perspective, what is the post-measurement state of the system?

Alice prepares a qubit in one of the three quantum signal states,

$$\begin{aligned} |\psi_0\rangle &= \quad |0\rangle\,, \\ |\psi_1\rangle &= -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\,, \\ |\psi_2\rangle &= -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\,, \end{aligned}$$

and sends it to Bob. Each state is chosen uniformly at random. Bob's task is to perform a measurement which allows him to maximise the probability of guessing correctly the signal state.

(e) [5 marks] It is known that in this case the optimal measurement is defined by the three operators $K_i = \sqrt{\frac{2}{3}}\,|\psi_i\rangle\langle\psi_i|$, where $i = 0, 1, 2$. Are they projectors? Are they Kraus operators? What is the probability of successful identification of the incoming signal state using this measurement?

(f) [4 marks] Bob couldn't perform the optimal measurement and instead decided to measure the qubit in the computational basis. What is the probability of obtaining the results 0 and 1 respectively? What can Bob infer about the signal states if the outcome is 1?

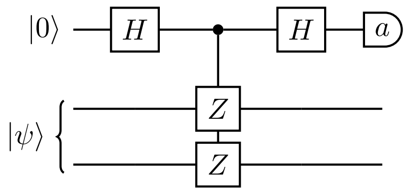
3. We say that operator S stabilises $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$.

(a) [5 marks] Consider the following two Bell states and show that

$$|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{is stabilised by} \quad \langle Z\otimes Z,\, X\otimes X\rangle,$$
$$|\bar{\Omega}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad \text{is stabilised by} \quad \langle Z\otimes Z,\, -X\otimes X\rangle,$$

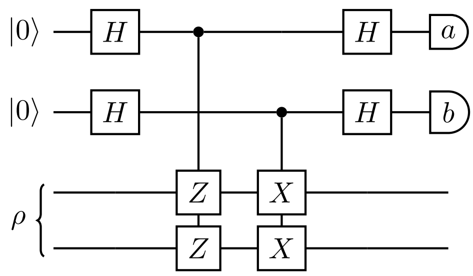
where $\langle Z\otimes Z,\, \pm X\otimes X\rangle$ stands for the stabiliser group generated by $Z\otimes Z$ and $\pm X\otimes X$. Which two-qubit state is stabilised by $\langle -Z\otimes Z,\ X\otimes X\rangle$, and which by $\langle -Z\otimes Z,\ -X\otimes X\rangle$?

The circuit below shows one way to measure the stabiliser $Z\otimes Z$ without destroying post-measurement states. The binary outcomes $a = 0$ and $a = 1$ correspond to Pauli eigenvalues $+1$ and -1, respectively.



(b) [6 marks] Step through the execution of this circuit, writing down the quantum states of the three qubits after each computational step, and show that it projects the input state $|\psi\rangle$ onto one of the two orthogonal subspaces associated with the two orthogonal projectors $\frac{1}{2}(\mathbb{1}\otimes\mathbb{1} \pm Z\otimes Z)$. Express these two projectors in terms of the projectors on the computational basis states, i.e. in terms of $|00\rangle\langle 00|$, $|01\rangle\langle 01|$, $|10\rangle\langle 10|$ and $|11\rangle\langle 11|$.

Observables $Z\otimes Z$ and $X\otimes X$, are measured on two qubits prepared in some, possibly mixed, input state ρ, as shown in the circuit diagram below



(c) [4 marks] What are the possible outcomes ab and what are the post-measurement states of the two qubits corresponding to these outcomes? Does it matter which observable is measured first?

(d) [4 marks] Let the input state be described by the density matrix

$$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|\,. \tag{1}$$

Show that this density matrix may also correspond to the scenario where the input state is randomly chosen with equal probability to be either $|\Omega\rangle$ or $|\bar{\Omega}\rangle$.

(e) [6 marks] You are provided with a source that was designed to repeatedly generate pairs of qubits in state $|\Omega\rangle$ but you are told that, due to decoherence, the source may, with some probability p, generate state ρ (as in Eq. (1)). How would you use the circuit above to estimate the fraction p of the decohered states?