INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Problem Sheet 2
Hilary Term

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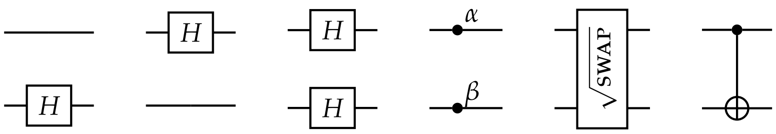
Questions Label: A - Bookwork B - Standard C - Challenging/Optional

2.1.B **Two-qubit operations.** The circuits below show six unitary operations on two qubits,

$P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

The square root of SWAP matrix has something in common with the square root of NOT. Start with writing the SWAP matrix.



The first four are described, respectively, by 4×4 unitary matrices which are tensor products $\mathbb{1} \otimes H$, $H \otimes \mathbb{1}$, $H \otimes H$ and $P(\alpha) \otimes P(\beta)$. The matrices of the two remaining gates, known as the square root of SWAP and controlled-NOT, stand out as they do not admit a tensor product decomposition in terms of single-qubit operations. Use the standard tensor product basis, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, and write down unitary matrices for each of the six gates.

Solution: Note:

- If not explicitly given, a matrix element is zero.
- Recall the standard notation $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\mathbb{1} \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix};$$

$$H \otimes \mathbb{1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix};$$

$$H \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

$$P(\alpha) \otimes P(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & e^{i\beta} & & \\ & & e^{i\alpha} & \\ & & & e^{i\alpha+i\beta} \end{pmatrix};$$

$$\sqrt{\text{SWAP}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\sqrt{NOT}} & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \frac{1+i}{2} & \frac{1-i}{2} & \\ & \frac{1-i}{2} & \frac{1+i}{2} & \\ & & & 1 \end{pmatrix}; \qquad CNOT = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

2.2.B **Basic entanglement.** Prove that the state of two qubits $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is entangled iff $ad - bc \neq 0$. Deduce that the state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + (-1)^k|11\rangle)$ is entangled for $k = 1$ and unentangled for $k = 0$. Express the latter case explicitly as a product state.

Solution: We start by decomposing $|\psi\rangle$ by pulling out the first qubit:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = |0\rangle(a|0\rangle + b|1\rangle) + |1\rangle(c|0\rangle + d|1\rangle)$$

The state ψ will be separable when the state of the second qubit on the two brackets is the same, i.e. if and only if $(a|0\rangle + b|1\rangle) \propto (c|0\rangle + d|1\rangle)$. This condition is equivalent to

$$\frac{a}{b} = \frac{c}{d} \Rightarrow ad - bc = 0$$

The reason why this is similar to the form of a determinant for a 2-by-2 matrix can be seen from the Schmidt decomposition in Sec. 5.10.2 of the online book.

For the particular case that $|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + (-1)^k|11\rangle\right)$, we have $a = b = c = 1/2$, $d = (-1)^k/2$, hence

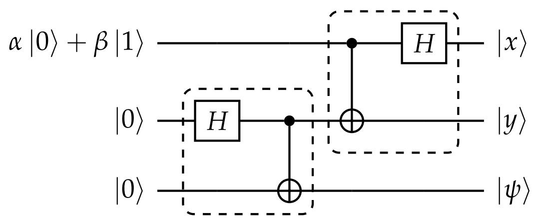
$$ad - bc = \frac{1}{4}\left((-1)^k - 1\right) = \begin{cases} 0, & k = 0 \Rightarrow \text{separable} \\ -1/2, & k = 1 \Rightarrow \text{entangled} \end{cases}$$

The separable state is ($k = 0$):

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}[|0\rangle(|0\rangle + |1\rangle) + |1\rangle(|0\rangle + |1\rangle)] \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

2.3.A **Quantum teleportation.** Consider the following quantum network (circuit), containing the Hadamard and the controlled-NOT gates,

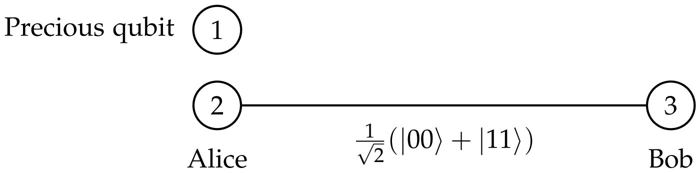
You should remember the action of the Hadamard and the controlled-NOT gates.



The measurement on the first two qubits (counting from the top) gives two binary digits, x and y. The third qubit is not measured. How does the state of the third qubit, $|\psi\rangle$, depend on the values x and y?

Divide et impera, that is, divide and conquer, a good approach to solving problems in mathematics (and in life). Start with smaller circuits, those surrounded by the dashed boxes.

Suppose the three qubits, which look very similar, are initially in a possession of an absent-minded Oxford student Alice. The first qubit is in a precious quantum state and this state is needed urgently for an experiment in Cambridge. Alice's colleague, Bob, pops in to collect the qubit. Once he is gone Alice realises that by mistake she gave him not the first but the third qubit, the one which is entangled with the second qubit (see the figure below).



The situation seems to be hopeless – Alice does not know the quantum state of the first qubit, Bob is now miles away and her communication with him is limited to at most one tweet. However, Alice and Bob are both very clever and attended the "Introduction to Quantum Information Science" course at Oxford. Can Alice rectify her mistake and save Cambridge science?

Solution: This is exactly Sec. 5.6.2 of the online book.

2.4.B **Partial traces and reduced density operators.** Consider two qubits in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle \otimes \left(\sqrt{\frac{2}{3}}\,|0\rangle + \sqrt{\frac{1}{3}}\,|1\rangle\right) + |0\rangle \otimes \left(\sqrt{\frac{2}{3}}\,|0\rangle - \sqrt{\frac{1}{3}}\,|1\rangle\right)\right]. \tag{1}$$

(1) What is the density operator ρ of the two qubits corresponding to state $|\psi\rangle$? Write it in the Dirac notation and explicitly as a matrix in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

(2) Find the reduced density operators ρ_1 and ρ_2 of the first and the second qubit, respectively. Again, write them in the Dirac notation and as matrices in the computational basis.

You obtain reduced density operators by taking partial traces, e.g. the partial trace over $\mathcal{H}_B$ is defined for the tensor product operators,
$\mathrm{Tr}_B\,(A \otimes B) = A\,(\mathrm{Tr}\,B)$
and extended to any other operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ by linearity. See the Prerequisite Material.

Solution:

(1) The state is:

$$|\psi\rangle = \frac{1}{\sqrt{3}}\,|00\rangle - \frac{1}{\sqrt{6}}\,|01\rangle + \frac{1}{\sqrt{3}}\,|10\rangle + \frac{1}{\sqrt{6}}\,|11\rangle \Rightarrow \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2}\\ -1\\ \sqrt{2}\\ 1\end{pmatrix}$$

The density matrix is:

$$\rho = |\psi\rangle\langle\psi| \Rightarrow \frac{1}{6}\begin{pmatrix}\sqrt{2}\\ -1\\ \sqrt{2}\\ 1\end{pmatrix}\begin{pmatrix}\sqrt{2} & -1 & \sqrt{2} & 1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2 & -\sqrt{2} & 2 & \sqrt{2}\\ -\sqrt{2} & 1 & -\sqrt{2} & -1\\ 2 & -\sqrt{2} & 2 & \sqrt{2}\\ \sqrt{2} & -1 & \sqrt{2} & 1\end{pmatrix}$$

(2) Trace over the first qubit:

$$\frac{1}{6}\begin{pmatrix}2 & -\sqrt{2} & 2 & \sqrt{2}\\ -\sqrt{2} & 1 & -\sqrt{2} & -1\\ 2 & -\sqrt{2} & 2 & \sqrt{2}\\ \sqrt{2} & -1 & \sqrt{2} & 1\end{pmatrix} \xRightarrow{\text{trace over the 1st qubit}} \frac{1}{6}\begin{pmatrix}4 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}\frac{2}{3} & 0\\ 0 & \frac{1}{3}\end{pmatrix} = \frac{2}{3}\,|0\rangle\langle 0| + \frac{1}{3}\,|1\rangle\langle 1|$$

Trace over the second qubit:

$$\frac{1}{6}\begin{pmatrix}2 & -\sqrt{2} & 2 & \sqrt{2}\\ -\sqrt{2} & 1 & -\sqrt{2} & -1\\ 2 & -\sqrt{2} & 2 & \sqrt{2}\\ \sqrt{2} & -1 & \sqrt{2} & 1\end{pmatrix} \xRightarrow{\text{trace over the 2nd qubit}} \frac{1}{6}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{2}\end{pmatrix} = \frac{2}{3}\,|+\rangle\langle +| + \frac{1}{3}\,|-\rangle\langle -|$$

2.5.B **Trace distance.** The trace norm of a matrix A is defined as

$$\|A\|_{tr} = \text{Tr}\left(\sqrt{A^\dagger A}\right).$$

The *positive* square root $\sqrt{\ }$ here means obtaining the *positive* operator that square to $A^\dagger A$.

Some Alternative Notations: $\sqrt{A^\dagger A}$ is often also denoted simply as $|A|$. The trace norm is often also call the *Schatten 1-norm* and denoted as $\| \ \|_1$. Hence, we can also use the following notations: $\|A\|_{tr} \equiv \|A\|_1 = \text{Tr}\left(\sqrt{A^\dagger A}\right) \equiv \text{Tr}\,|A|$.

(1) Show that the trace norm of any self-adjoint matrix is the sum of the absolute values of its eigenvalues. What is the trace norm of a density matrix?

(2) The trace distance between density matrices ρ_1 and ρ_2 is defined as

$$d(\rho_1, \rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|_{tr}.$$

What is the trace distance between two pure states $|\phi\rangle$ and $|\psi\rangle$?

Solution:

(1) By the spectral decomposition, we have:

$$A = \sum_i \lambda_i |i\rangle\langle i|$$
$$A^\dagger A = \sum_i |\lambda_i|^2 |i\rangle\langle i|$$
$$\sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle\langle i|$$

The trace of an operator is the sum of the operator's eigenvalues, thus the trace norm of A is $\|A\|_{tr} = \sum_i |\lambda_i|$. A density matrix only has positive eigenvalues, and it has trace 1, so the sum of the absolute value of the eigenvalues is 1.

See Lecture 4.8.

(2) Using $|\psi\rangle$ as basis, we have $|\phi\rangle = \alpha|\psi\rangle + \beta|\psi_\perp\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. The matrix representation of $\rho_1 - \rho_2$ is then:

$$|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} |\alpha|^2 & \beta^*\alpha \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} = \begin{pmatrix} |\beta|^2 & -\beta^*\alpha \\ -\alpha^*\beta & -|\beta|^2 \end{pmatrix}$$

Given that

$$\lambda_1 + \lambda_2 = \text{Tr}(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|) = 0$$
$$\lambda_1\lambda_2 = \det(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|) = -|\beta|^2\left(|\alpha|^2 + |\beta|^2\right) = -|\beta|^2$$

we have $\lambda = \pm|\beta|$. Hence, the trace distance is

$$d(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \frac{|\lambda_1| + |\lambda_2|}{2} = |\beta| = \sqrt{1 - |\alpha|^2} = \sqrt{1 - |\langle\phi|\psi\rangle|^2}$$

2.6.B **How well can we distinguish two quantum states?.** We have a physical system is equally likely to be output the state ρ_1 or the state ρ_2, and we want to tell which state it has produced using a single measurement.

(1) Suppose ρ_1 and ρ_2 commute. Use the spectral decomposition of ρ_1 and ρ_2 in their common eigenbasis and describe the optimal measurement that can distinguish between the two states. Using this optimal measurement, show that the probability of successfully distinguishing between ρ_1 and ρ_2 using a single measurement is at most:

This special case is essentially a classical problem of differentiating between two probability distributions.

$$P(\text{succ}) = \frac{1}{2}\left[1 + d(\rho_1, \rho_2)\right]. \tag{2}$$

where d is the trace distance described in Question 2.5.

(2) In fact, Eq. (2) is true even if ρ_1 and ρ_2 do not commute. Using this, now suppose you are given one of the two, randomly selected, qubits of state $|\psi\rangle$ in Eq. (1), what is the maximal probability with which you can determine whether it is the first or the second qubit?

Solution:

(1) If ρ_1 and ρ_2 commute, they can be simultaneously diagonalisable:

$$\rho_1 = \sum_i p_i |i\rangle\langle i| \qquad \rho_2 = \sum_i q_i |i\rangle\langle i|$$

Hence, their trace distance is simply

$$d(\rho_1,\rho_2) = \frac{\mathrm{Tr}\,|\rho_1-\rho_2|}{2} = \frac{\sum_i |p_i - q_i|}{2}$$

Now suppose we are randomly given one of $\{\rho_1,\rho_2\}$ and we want to successful guess which is it by performing a measurement on it. The optimal strategy would be performing a projective measurement onto the set of basis $\{|i\rangle\}$. When the output is i, then the state is more likely to be ρ_1 if $p_i \geq q_i$ and conversely the state is more likely to be ρ_2 if $p_i < q_i$.

Following such a strategy, if the actual underlying state is ρ_1, then we will have the right guess whenever $p_i \geq q_i$, thus the total probability of successfully guessing the right state in this case is then:

$$P(succ|\rho_1) = \sum_{p_i \geq q_i} p_i$$

Similarly, we have:

$$P(succ|\rho_2) = \sum_{q_i > p_i} q_i$$

Since we have equal probability of choosing ρ_1 and ρ_2: $P(\rho_1) = P(\rho_2) = \frac{1}{2}$, the total success probability is:

$$\begin{aligned} P(succ) &= P(succ|\rho_1)P(\rho_1) + P(succ|\rho_2)P(\rho_2) \\ &= \frac{1}{2}\sum_{p_i\geq q_i} p_i + \frac{1}{2}\sum_{q_i>p_i} q_i = \frac{1}{2}\sum_i \max(p_i,q_i). \end{aligned}$$

Since $\max(p_i,q_i) = \frac{1}{2}(p_i + q_i + |p_i - q_i|)$, we have

$$P(succ) = \frac{1}{2}\sum_i \max(p_i,q_i) = \frac{1}{2}\left(1 + \frac{\sum_i |p_i - q_i|}{2}\right) = \frac{1 + d(\rho_1,\rho_2)}{2}$$

i.e. the trace distance is proportional to the maximum success probability that we can distinguish between ρ_1 and ρ_2. This is true even when ρ_1 and ρ_2 are not commuting, but in that case, we need to generalise projective measurement to POVM.

(2)

$$\rho_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} \qquad \rho_2 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$\rho_1 - \rho_2 = \frac{1}{6}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Hence:

$$\lambda_1 + \lambda_2 = \mathrm{Tr}(\rho_1 - \rho_2) = 0$$

$$\lambda_1\lambda_2 = \det(\rho_1 - \rho_2) = -\frac{1}{18}$$

which means:

$$\lambda = \pm\frac{1}{\sqrt{18}} = \pm\frac{1}{3\sqrt{2}}$$

Hence,

$$d(\rho_1, \rho_2) = \frac{\mathrm{Tr}\,|\rho_1 - \rho_2|}{2} = \frac{\sum_i |\lambda_i|}{2} = \frac{1}{3\sqrt{2}}.$$

and

$$P(succ) = \frac{1 + d(\rho_1, \rho_2)}{2} = \frac{1}{2} + \frac{1}{6\sqrt{2}} \approx 0.6.$$

2.7.B **Bloch vectors.** Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(1) Check that such parametrised ϱ has all the mathematical properties of a density matrix as long as the length of the Bloch vector does not exceed 1.

(2) Draw the Bloch sphere and mark all the convex combinations of states $|0\rangle$ and $|1\rangle$, i.e. the states of the form

$$\rho = p_0\,|0\rangle\langle 0| + p_1\,|1\rangle\langle 1|,$$

where p_0 and p_1 are non-negative and $p_0 + p_1 = 1$. How would you generate such states?

(3) Draw the Bloch sphere and mark the Pauli eigenstates and all the convex combinations of the Pauli eigenstates.

(4) A qubit in state $|0\rangle$ is modified by a long sequence of randomly selected Clifford gates. You remember the sequence at first, but as time passes you are less and less certain what it was, until you completely forget it. Explain why, from your perspective, the final state of the qubit has a Bloch vector that lies somewhere inside the octahedron with vertices representing the six eigenstates of the Pauli operators X, Y, and Z. Where is this Bloch vector when you still remember the Clifford sequence, and where is it when you have completely forgotten the sequence?

Remember from Question 1.6 that a Clifford gate will map any Pauli eigenstate to another Pauli eigenstate.

(5) Two qubits are in quantum states described by their respective Bloch vectors, $\vec{s}_1$ and $\vec{s}_2$. What is the trace distance between the two quantum states?

Solution:

(1) Properties of Density Matrix:

- Unit Trace: Using $\mathrm{Tr}(\sigma_i) = 0$ and $\mathrm{Tr}(\mathbb{1}) = 2$, we have $\mathrm{Tr}(\rho) = 1$.
- Positivity: We can explicitly verify the the eigenvalues of $\vec{s}\cdot\vec{\sigma}$ is $\pm|\vec{s}|$. Hence, the eigenvalues of $\rho = \frac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right)$ are simply $\lambda_\pm = \frac{1}{2}\left(1 \pm |\vec{s}|\right)$. Since $|\vec{s}| \leq 1$, we have $\lambda_\pm \geq 0$ and thus ρ is positive.

  Alternatively, we know that we can use some unitary transformation U to rotate the Bloch vector $\vec{s}$ such that it becomes another Bloch vector $\vec{s}' = (|\vec{s}|, 0, 0)$ that aligns with the z-axis:

  $$U\vec{s}\cdot\vec{\sigma}U^\dagger = \vec{s}'\cdot\vec{\sigma} = |\vec{s}|\sigma_z$$

  Since unitary transformation will not change the eigenvalues, we know that $\vec{s}\cdot\vec{\sigma}$ and $|\vec{s}|\sigma_z$ have the same eigenvalues, which are $\pm|\vec{s}|$. The rest follows similarly

(2) $\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1|$ lives on the green line in Fig. 1. It can be generated by preparing $|0\rangle$ with probability p_0 and preparing $|1\rangle$ with probability p_1.

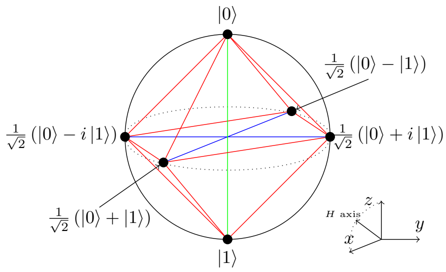


FIGURE 1. Bloch Sphere.

(3) The Pauli eigenstates are the black dots in Fig. 1. Their convex combinations are just all the states living within the red octahedron in Fig. 1.

(4) A Clifford operator will map a Pauli eigenstate into another Pauli eigenstate. Since Clifford operations forms a group, a sequence of Clifford operation can be view as one single Clifford operation as a whole. Since we forget part of the sequence, the overall Clifford operation can be one of the many possible Clifford operations, which will map $|0\rangle$ into one of the 6 possible Pauli eigenstate with a certain probability distribution. Hence, our resultant state is a some convex combination of the Pauli eigenstates, which as discussed in (b) is a state living within the red octahedron in Fig. 1.

If we exactly remember the circuit, we know exactly what is the overall Clifford operation, which enable us to know exactly what is the output Pauli eigenstate. Hence, the output state is one of the Pauli eigenstates, which are the *vertices of the octahedron*.

If we completely forgot the circuit and the circuit is long enough, essentially the overall circuit can be any Clifford operation with uniform probability. Hence, the resultant state is a uniform sum of of all possible Pauli eigenstates, resulting in a maximally mixed states that lives at the *centre of the Bloch sphere*.

(5) The trace distance between two density matrix is:

$$d(\rho_1, \rho_2) = \frac{\mathrm{Tr}\,|\rho_1 - \rho_2|}{2} = \frac{\mathrm{Tr}\,|(\vec{r}_1 - \vec{r}_2) \cdot \vec{\sigma}|}{4}$$

Following similar arguments in question (1), the eigenvalues of $(\vec{s}_1 - \vec{s}_2) \cdot \vec{\sigma}$ are $\pm|\vec{s}_1 - \vec{s}_2|$. Remember that $\mathrm{Tr}\,|(\vec{s}_1 - \vec{s}_2) \cdot \vec{\sigma}|$ is simply the sum of the absolute value of these two eigenvalues:

$$d(\rho_1, \rho_2) = \frac{|\vec{s}_1 - \vec{s}_2|}{2}.$$

Hence, the trace distance between two single-qubit states is simply half of the Euclidean distance between them on the Bloch sphere.