INTRODUCTION TO QUANTUM INFORMATION SCIENCE **Problem Sheet 4**
Hilary Term

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Questions Label: A - Bookwork B - Standard C - Challenging/Optional

4.1.B **Completely positive maps.** Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger,$$

where matrices A_k satisfy $\sum_k A_k^\dagger A_k = \mathbb{1}$.

(1) Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.

(2) A qubit in state ϱ is transmitted through a depolarising channel that effects a completely positive map

$$\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x \varrho \sigma_x + \sigma_y \varrho \sigma_y + \sigma_z \varrho \sigma_z\right),$$

for some $0 \leq p \leq 1$. Show that under this map the Bloch vector associated with ϱ shrinks by the factor $(3-4p)/3$.

Solution:

(1)
- Positivity preserving:

  Here we simply want to prove $\mathcal{A}$ preserves positivity instead of proving its complete positivity. Since $\rho \geq 0$, we have $\rho = MM^\dagger$. Hence, $A_k \rho A_k^\dagger = (A_k M)(A_k M)^\dagger \geq 0$, which means $\sum_k A_k \rho A_k^\dagger \geq 0$.

- Trace preserving:

$$\mathrm{Tr}(\mathcal{A}(\rho)) = \mathrm{Tr}\left(\sum_k A_k \rho A_k^\dagger\right) = \mathrm{Tr}\left(\sum_k A_k^\dagger A_k \rho\right) = \mathrm{Tr}(\rho)$$

- Composing two CP map gives another CP map:

$$\mathcal{A}(\mathcal{B}(\rho)) = \sum_{j,k} A_j B_k \rho B_k^\dagger A_j^\dagger = \sum_{j,k} C_{jk} \rho C_{jk}^\dagger$$

  with

$$C_{jk} = A_j B_k.$$

  We also have

$$\sum_{j,k} C_{jk}^\dagger C_{jk} = \sum_k B_k^\dagger \left(\sum_j A_j^\dagger A_j\right) B_k = I.$$

(2) It is possible to do it via direct explicit calculation. The arguments below will give us a bit more perspective into the channel.

Rewrite the depolarising channel as:

$$\mathcal{D}_q = (1-q)\mathcal{I} + \frac{q}{4}(\mathcal{I} + \mathcal{X} + \mathcal{Y} + \mathcal{Z}) = (1-q)\mathcal{I} + q\mathcal{D}_1$$

where $q = \frac{4}{3}p$. Here $\mathcal{D}_1 = \frac{1}{4}(\mathcal{I} + \mathcal{X} + \mathcal{Y} + \mathcal{Z})$ is simply the completely depolarising channel that will completely randomised any input single-qubit state and turn them into the maximally mixed state:

$$\mathcal{D}_1(\rho) = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z) = \frac{I}{2}.$$

Hence, when the depolarising channel act on the state ρ, we have:

$$\mathcal{D}_q(\rho) = (1-q)\rho + q\mathcal{D}_1(\rho) = (1-q)\rho + q\frac{I}{2}.$$

i.e. the resultant state is a probabilistic mixture of $1-q$ probability of the original state (Bloch vector $\vec{s}$) and q probability of the maximally mixed state (Bloch vector $\vec{0}$). Hence, the resultant state simply have the Bloch vector $(1-q)\vec{s}$.

Returning to our question, we have $q = \frac{4}{3}p$. Hence, the Bloch vector is shrunk by a factor of $1 - q = 1 - \frac{4}{3}p$.

4.2.B **Positive but not completely positive maps.** Consider a map $\mathcal{N}$, called universal-NOT, which acts on single qubit density matrices and is defined by its action on the identity and the three Pauli matrices

Any 2×2 matrix can be written as a linear composition of the identity and the three Pauli matrices as discussed in Question 1.1.

$$\mathcal{N}(\mathbb{1}) = \mathbb{1} \quad \mathcal{N}(\sigma_x) = -\sigma_x \quad \mathcal{N}(\sigma_y) = -\sigma_y \quad \mathcal{N}(\sigma_z) = -\sigma_z$$

(1) Describe the action of this map in terms of the Bloch vectors.

(2) Explain why $\mathcal{N}$, acting on a single qubit, maps density matrices to density matrices.

(3) The joint state of two qubits is described by the density matrix

$$\rho = \tfrac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\right),$$

Apply $\mathcal{N}$ to the first qubit leaving the second qubit intact. Write the resulting matrix and explain why $\mathcal{N}$ is not a completely-positive map.

Solution:

(1) We have

$$\rho = \frac{\mathbb{1} + \vec{s}\cdot\vec{\sigma}}{2} \Rightarrow \mathcal{N}(\rho) = \frac{I - \vec{s}\cdot\vec{\sigma}}{2}$$

It maps a Bloch vector $\vec{s}$ to $-\vec{s}$.

(2) If $\vec{s}$ is a valid Bloch vector that satisfy $|\vec{s}| \leq 1$, then $-\vec{s}$ is also a valid Bloch vector.

(3) To show $\mathcal{N}$ is completely positive we need to show that its corresponding Choi matrix is positive. To show that $\mathcal{N}$ is *not* completely positive, we only need to find an input state such that after it pass through $\mathcal{N}$, the output matrix is *not* positive and thus is not a valid density matrix. Such an input state is given by the problem and the corresponding output state is:

$$\mathcal{N}(\rho) = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z\right).$$

It is easy to verify that the eigenstates of $\mathcal{N}(\rho)$ are the four Bell states, within which the eigenvalue of $|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ can be obtained by

$$\begin{aligned}\mathcal{N}(\rho)\,|\Psi_{00}\rangle &= \frac{1}{4\sqrt{2}}\left(\mathbb{1}\otimes\mathbb{1} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_z\right)(|00\rangle + |11\rangle)\\ &= \frac{1}{4}\left(|\Psi_{00}\rangle - |\Psi_{00}\rangle - |\Psi_{00}\rangle - |\Psi_{00}\rangle\right)\\ &= -\frac{1}{2}\,|\Psi_{00}\rangle\,.\end{aligned}$$

Since the eigenvalue is negative, $\mathcal{N}(\rho)$ is not positive and thus $\mathcal{N}$ is not completely positive.

4.3.B **Approximate cloning.** Consider a hypothetical universal quantum cloner that operates on two qubits and on some auxiliary system. Given one qubit in any quantum state $|\psi\rangle$ and the other one in a prescribed state $|0\rangle$ it maps

$$|\psi\rangle\,|0\rangle\,|R\rangle \mapsto |\psi\rangle\,|\psi\rangle\,|R'\rangle\,,$$

where $|R\rangle$ and $|R'\rangle$ are, respectively, the initial and the final state of any other auxiliary system that may participate in the cloning process ($|R'\rangle$ may depend on $|\psi\rangle$).

(1) Show that such a cloner is impossible.

But supposed we are willing to settle for an imperfect copy? It turns out that the best approximation to the universal quantum cloner is the following transformation

$$|\psi\rangle\,|0\rangle\,|0\rangle \mapsto \sqrt{\frac{2}{3}}\,|\psi\rangle\,|\psi\rangle\,|\psi\rangle + \sqrt{\frac{1}{6}}\left(|\psi\rangle\left|\psi^{\perp}\right\rangle + \left|\psi^{\perp}\right\rangle|\psi\rangle\right)\left|\psi^{\perp}\right\rangle$$

where $\left|\psi^{\perp}\right\rangle$ is a normalised state vector orthogonal to $|\psi\rangle$ and the auxiliary system is another qubit.

(2) Given the transformation above explain why the reduced density matrices of the first and the second qubit must be identical after the transformation.

(3) Show that the reduced density matrix of the first (and the second) qubit can be written as

$$\rho = \frac{5}{6}\,|\psi\rangle\!\langle\psi| + \frac{1}{6}\left|\psi^{\perp}\right\rangle\!\left\langle\psi^{\perp}\right|.$$

(4) What is the probability that the clone in state ρ will pass a test for being in the original state $|\psi\rangle$?

(5) What is the relation between the Bloch vectors of $|\psi\rangle\!\langle\psi|$ and ρ?

Solution:

(1) Remember that any quantum channel on a subsystem can be modelled using a unitary operation on the extended quantum system including the environment. Since here we have explicitly include the environment with the initial state $|R\rangle$, we can write the cloner as a unitary operator U that performs:

$$U\,|\psi\rangle\,|0\rangle\,|R\rangle = |\psi\rangle\,|\psi\rangle\,|R'\rangle$$

To clone another state $|\phi\rangle$ using the same cloner (since it is universal), we have:

$$U\,|\phi\rangle\,|0\rangle\,|R\rangle = |\phi\rangle\,|\phi\rangle\,|R''\rangle$$

Taking the inner product between the two equation we have:

$$\langle R| \langle 0| \langle \phi| U^\dagger U |\psi\rangle |0\rangle |R\rangle = \langle R''| \langle \phi| \langle \phi|\psi\rangle |\psi\rangle |R'\rangle$$
$$\langle \phi|\psi\rangle = \langle \phi|\psi\rangle^2 \langle R''|R'\rangle$$

which means that $\langle \phi|\psi\rangle = \langle R''|R'\rangle = 1$ or $\langle \phi|\psi\rangle = 0$, which does not hold for any two arbitrary state $|\psi\rangle$ and $|\phi\rangle$.

(2)

$$|\Psi\rangle = \sqrt{\frac{2}{3}} |\psi\rangle |\psi\rangle |\psi\rangle + \sqrt{\frac{1}{6}} \left(|\psi\rangle \left|\psi^\perp\right\rangle + \left|\psi^\perp\right\rangle |\psi\rangle \right) \left|\psi^\perp\right\rangle$$

The state remains unchanged under the exchange of the first two qubits, thus their reduced density matrix would be the same.

(3) Tracing out the *third* qubit we simply have:

$$\frac{2}{3} |\psi\psi\rangle\langle\psi\psi| + \frac{1}{6} \left(\left|\psi\psi^\perp\right\rangle + \left|\psi^\perp\psi\right\rangle \right) \left(\left\langle\psi\psi^\perp\right| + \left\langle\psi^\perp\psi\right| \right),$$

Then tracing out the *second* qubit we have:

$$\frac{2}{3} |\psi\rangle\langle\psi| + \frac{1}{6} |\psi\rangle\langle\psi| + \frac{1}{6} \left|\psi^\perp\right\rangle\left\langle\psi^\perp\right| = \frac{5}{6} |\psi\rangle\langle\psi| + \frac{1}{6} \left|\psi^\perp\right\rangle\left\langle\psi^\perp\right|$$

(4) The probability that the cloned state pass the test is $\frac{5}{6}$.

(5) The Bloch vector is in the same direction, but the magnitude is shrunk by a factor of $\frac{5}{6} - \frac{1}{6} = \frac{2}{3}$.

4.4.B **CP maps revisited.** Any linear transformation (superoperator) T acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|a\rangle\langle b|$, where $a, b = 0, 1$, and can be represented as a 4×4 matrix,

$$\tilde{T} = \left[\begin{array}{c|c} T(|0\rangle\langle 0|) & T(|0\rangle\langle 1|) \\ \hline T(|1\rangle\langle 0|) & T(|1\rangle\langle 1|) \end{array} \right].$$

Write down $\tilde{T}$ for:

(1) transposition, $\varrho \mapsto \varrho^T$,

(2) depolarising channel, $\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x \varrho \sigma_x + \sigma_y \varrho \sigma_y + \sigma_z \varrho \sigma_z\right)$, for some $0 \le p \le 1$.

Show that for completely positive maps T matrix $\tilde{T}$ must be positive semidefinite.

Solution:

(1)

$$\widetilde{T} = \left(\begin{array}{c|c} |0\rangle\langle 0| & |1\rangle\langle 0| \\ \hline |0\rangle\langle 1| & |1\rangle\langle 1| \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

(2)

$$\widetilde{T} = \left(\begin{array}{cc|cc} 1-\frac{2}{3}p & 0 & 0 & 1-\frac{4}{3}p \\ 0 & \frac{2}{3}p & 0 & 0 \\ \hline 0 & 0 & \frac{2}{3}p & 0 \\ 1-\frac{4}{3}p & 0 & 0 & 1-\frac{2}{3}p \end{array} \right)$$

(3) Recall that any completely positive map can be written in terms of Kraus representation, thus T can be written as:

$$T(\rho) = \sum_i A_i \rho A_i^\dagger \quad \text{with } \sum_i A_i^\dagger A_i = I.$$

Hence,

$$\begin{aligned}
\widetilde{T} &= \left(\begin{array}{c|c} \sum_i A_i |0\rangle \langle 0| A_i^\dagger & \sum_i A_i |0\rangle \langle 1| A_i^\dagger \\ \hline \sum_i A_i |1\rangle \langle 0| A_i^\dagger & \sum_i A_i |1\rangle \langle 1| A_i^\dagger \end{array} \right) \\
&= \sum_i \left(\begin{array}{c|c} A_i |0\rangle \langle 0| A_i^\dagger & A_i |0\rangle \langle 1| A_i^\dagger \\ \hline A_i |1\rangle \langle 0| A_i^\dagger & A_i |1\rangle \langle 1| A_i^\dagger \end{array} \right) \\
&= \sum_i \left(\begin{array}{c|c} A_i |0\rangle & 0 \\ \hline A_i |1\rangle & 0 \end{array} \right) \left(\begin{array}{c|c} \langle 0| A_i^\dagger & \langle 1| A_i^\dagger \\ \hline 0 & 0 \end{array} \right) \\
&= \sum_i M_i M_i^\dagger
\end{aligned}$$

with

$$M_i = \left(\begin{array}{c|c} A_i |0\rangle & 0 \\ \hline A_i |1\rangle & 0 \end{array} \right).$$

Hence, T is completely positive implies that $\widetilde{T}$ is positive semi-definite.

On the other hand, is the converse true? That is can we say that if $\widetilde{T}$ is positive (semi-definite), then T is completely positive? Recall that we can prove that a channel T is completely positive by proving its Choi matrix (denoted as T_{ch}) is positive, where the Choi matrix of a *single-qubit* channel T is simply the resultant density matrix after apply T to one of the qubits in the bell pair $|\Omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$:

$$\begin{aligned}
T_{ch} &= (I \otimes T) |\Omega\rangle \langle \Omega| \\
&= \frac{1}{2} (I \otimes T) \sum_{i,j=0}^{1} |ii\rangle \langle jj| \\
&= \frac{1}{2} \sum_{i,j=0}^{1} |i\rangle \langle j| \otimes T(|i\rangle \langle j|) \\
&= \frac{1}{2} \left(\begin{array}{c|c} T(|0\rangle \langle 0|) & T(|0\rangle \langle 1|) \\ \hline T(|1\rangle \langle 0|) & T(|1\rangle \langle 1|) \end{array} \right) \\
&= \frac{1}{2} \widetilde{T}
\end{aligned}$$

Hence, if $\widetilde{T}$ is positive, then the corresponding Choi matrix T_{ch} is positive and the corresponding channel T is completely positive.

4.5.B **Quantum error correction.**

(1) Draw a quantum network (circuit) that encodes a single qubit state $\alpha|0\rangle + \beta|1\rangle$ into the state $\alpha|00\rangle + \beta|11\rangle$ of two qubits. Here and in the following α and β are some unknown generic complex coefficients.

(2) Two qubits were prepared in state $\alpha|00\rangle + \beta|11\rangle$, exposed to bit flip-errors, and then measured with an ancillary qubit, as shown in Fig. 1. The result of the measurement is x. Can you infer the absence of errors when $x = 0$? Can you infer the presence of errors when $x = 1$? Can you correct any detected errors?

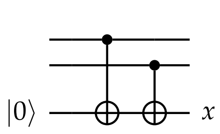


Fig.1

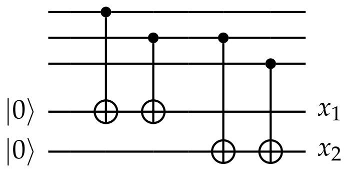


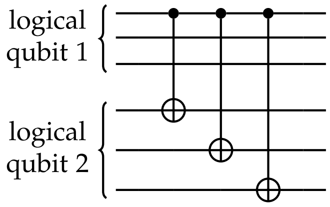
Fig.2

Three qubits were prepared in state $\alpha|000\rangle + \beta|111\rangle$ and then, by mistake, someone applied the Hadamard gate to one of them, but nobody remembers which one. Your task is to recover the original state of the three qubits.

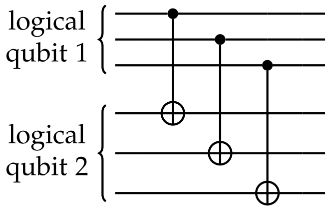
(3) Express the Hadamard gate as the sum of two Pauli matrices. Pick up one of the three qubits and apply the Hadamard gate. How is the state $\alpha|000\rangle + \beta|111\rangle$ modified? Interpret this in terms of bit-flip and phase-flip errors.

(4) You perform the error syndrome measurement shown in Fig. 2. Suppose the outcome of the measurement is $x_1 = 0, x_2 = 1$. How would you recover the original state? Describe the recovery procedure when $x_1 = 0, x_2 = 0$.

The figure below shows two implementations of a controlled-NOT gate acting on the encoded states of the three qubit code.



Implementation A

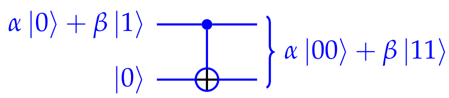


Implementation B

(5) Assume that the only sources of errors are individual controlled-NOT gates which produce bit-flip errors in their outputs. These errors are independent and occur with a small probability p. For each of the two implementations find the probability of generating unrecoverable errors at the output. Which of the two implementations is fault-tolerant?

Solution:

(1) The circuit is



(2)
- No Errors:

$$(\alpha|00\rangle + \beta|11\rangle)|0\rangle \mapsto (\alpha|00\rangle + \beta|11\rangle)|0\rangle$$

- Error on qubit 1:

$$(\alpha|10\rangle + \beta|01\rangle)|0\rangle \mapsto (\alpha|10\rangle + \beta|01\rangle)|1\rangle$$

- Error on qubit 2:

$$(\alpha|01\rangle + \beta|10\rangle)|0\rangle \mapsto (\alpha|01\rangle + \beta|10\rangle)|1\rangle$$

- Two Errors:

$$(\alpha |11\rangle + \beta |00\rangle) |0\rangle \mapsto (\alpha |11\rangle + \beta |00\rangle) |0\rangle$$

Outcome $x = 0$ corresponds to the case of no errors or two errors, thus we cannot conclude that there is no errors.

Outcome $x = 1$ corresponds to the case of a single error on qubit 1 or 2, thus we can conclude that there is an error occurring. However, we cannot correct it since we do not know which qubit is flipped.

(3) The noiseless state is:

$$|\psi\rangle = \alpha |000\rangle + \beta |111\rangle$$

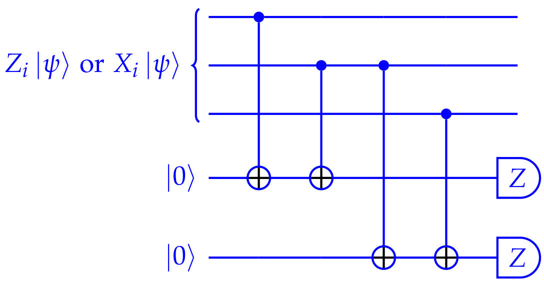
When a random Hadamard error occur to the ith qubit, we have:

$$H_i |\psi\rangle = \frac{1}{\sqrt{2}}(X_i + Z_i) |\psi\rangle = \frac{1}{\sqrt{2}}(\underbrace{X_i |\psi\rangle}_{i\text{th qubit flipped}} + \underbrace{Z_i |\psi\rangle}_{\text{phase flipped}})$$

where

$$\begin{aligned} Z_i |\psi\rangle &= \alpha |000\rangle - \beta |111\rangle \quad \forall i \\ X_1 |\psi\rangle &= \alpha |100\rangle + \beta |011\rangle \\ X_2 |\psi\rangle &= \alpha |010\rangle + \beta |101\rangle \\ X_3 |\psi\rangle &= \alpha |001\rangle + \beta |110\rangle . \end{aligned}$$

(4) With the starting state above, stepping through the syndrome measurement circuit



we can obtain the corresponding measurement outcomes using the error propagation rule of CNOT:

$$\begin{aligned} Z_i |\psi\rangle &\Rightarrow \{0,0\} \\ X_1 |\psi\rangle &\Rightarrow \{1,0\} \\ X_2 |\psi\rangle &\Rightarrow \{1,1\} \\ X_3 |\psi\rangle &\Rightarrow \{0,1\}. \end{aligned}$$

Suppose the Hadamard error occur to the third qubit, the starting state $H_3 |\psi\rangle$ is a superposition of $Z_i |\psi\rangle$ and $X_3 |\psi\rangle$, going through the syndrome measurement circuit, we will either obtain

- Outcome $= \{0,0\}$, state collapse to $Z_i |\psi\rangle$, correction by applying an additional Z_i.
- Outcome $= \{0,1\}$, state collapse to $X_3 |\psi\rangle$, correction by applying an additional X_3.

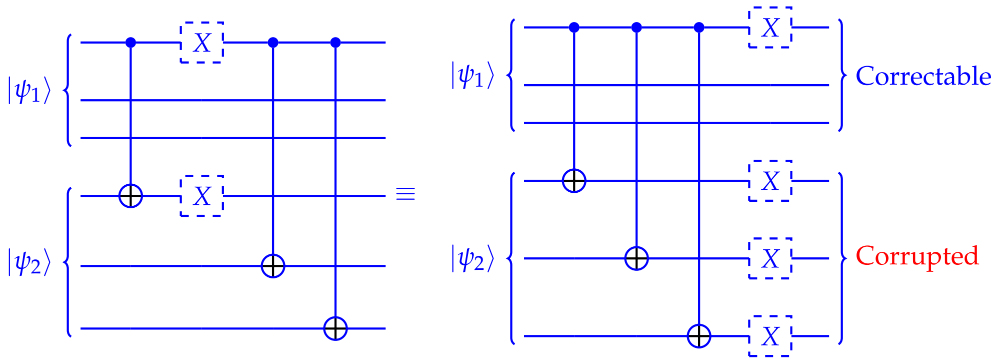
Similarly for the case of the Hadamard error occur to the other two qubits.

(5) Given two states encoded in the repetition code:

$$|\psi_1\rangle = \alpha_1 |000\rangle + \beta_1 |111\rangle$$
$$|\psi_2\rangle = \alpha_2 |000\rangle + \beta_2 |111\rangle$$

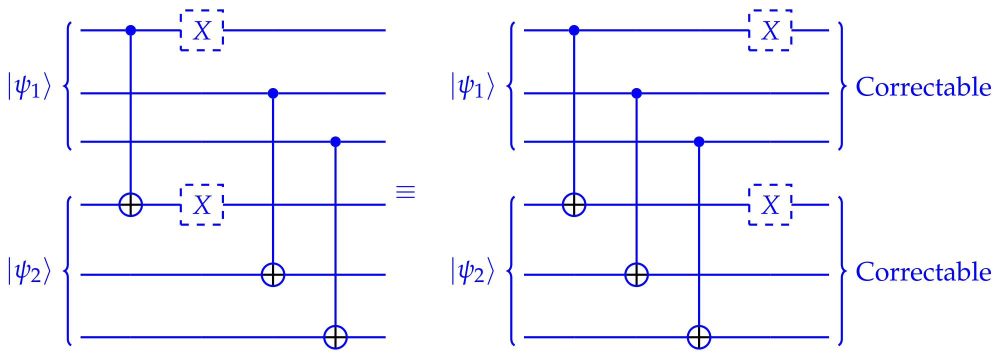
There are two two ways to apply logical CNOT between the *encoded information* of both logical qubits above, if the first physical CNOT fail (with probability p), the error will propagate in the following way:

- Implementation 1:



Hence, failure of one single *physical* component (the first CNOT) can lead to the loss of *logical* information (corruption of the second logical qubits). More specifically, when each individual physical CNOT fail with the probability p, the logical CNOT implemented in this way will also fail with the probability $\mathcal{O}(p)$.

- Implementation 2:



Hence, failure of one single *physical* component (physical CNOTs) will *not* lead to the loss of *logical* information. More specifically, when each individual physical CNOT fail with the probability p, the logical CNOT implemented in this way will only fail when two physical CNOT fails simultaneously, and thus will fail with the probability $\mathcal{O}(p^2)$.

4.6.B **Stabilisers define vectors and subspaces.**

In Problem sheet 1, we have discuss the concept of 1-qubit Pauli group and also the concept of stabiliser groups. Here we will further explore these concepts.

The n-qubit Pauli group is defined as

$$\mathbb{G}_n = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \otimes \{\pm 1, \pm i\}$$

where X, Y, Z are the Pauli matrices. Each element of $\mathbb{G}_n$ is, up to an overall phase ± 1, $\pm i$, a tensor product of Pauli matrices and identity matrices acting on the n qubits.

A unitary S stabilises $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$ and we have shown in Problem sheet 1 that the set of stabilisers of a given state $|\psi\rangle$ forms a group (known as the stabiliser group). As we will see later, we will generalise the concept of stabiliser groups from stabilising a state to stabilising a subspace (i.e. stabilising all states in the subspace), which is called a *code space*. We shall restrict our attention to stabiliser groups $\mathbb{S}$ that are subgroups of $\mathbb{G}_n$.

(1) Explain why in order to have a non-trivial (non-zero-dimension) code space, the stabiliser group must be Abelian (i.e. all of its elements commute) and do not contain the element $-\mathbb{1}$?

(2) Explain why all such stabilisers (except the identity $\mathbb{1}$) have trace zero and square to $\mathbb{1}$.

(3) Show that each stabiliser S has the same number of eigenvectors with eigenvalues $+1$ and -1, and hence "splits" the 2^{2n} dimensional Hilbert space in half. How would you describe the action of the two operators $\frac{1}{2}(\mathbb{1} \pm S)$?

(4) Consider two stabiliser generators, S_1 and S_2. Show that eigenvalue $+1$ subspace of S_1 is split again in half by S_2. That is, in that subspace exactly half of the S_2 eigenvectors have eigenvalue $+1$ and the other half -1.

Hint: Show that $\operatorname{tr} \frac{1}{2}(\mathbb{1} + S_1)S_2 = 0$.

(5) If a stabiliser group in the Hilbert space of dimension 2^n has a minimal number of generators, $S_1, \ldots, S_r$, what is dimension of the stabiliser subspace?

(6) State $|0\rangle$ is stabilised by Z and state $|1\rangle$ is stabilised by $-Z$. What are stabiliser generators for the standard basis of two qubits, i.e. for the states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$? What are stabiliser generators for each of the four Bell states?

(7) Construct stabiliser generators for an $n = 3$, $k = 1$ (n physical qubits encoding k logical qubits) code that can correct a single bit flip, i.e. ensure that recovery is possible for any of the errors in the set $\mathcal{E} = \{\mathbb{1}\mathbb{1}\mathbb{1}, X\mathbb{1}\mathbb{1}, \mathbb{1}X\mathbb{1}, \mathbb{1}\mathbb{1}X\}$. Find an orthonormal basis for the two-dimensional code subspace.

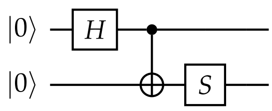
We often drop the tensor product symbol, e.g. $\mathbb{1} \otimes X \otimes \mathbb{1} \equiv \mathbb{1}X\mathbb{1}$. For commonly used single-qubit gates, sometimes we simply use subscripts to denote which qubits they acts on, e.g. $\mathbb{1} \otimes X \otimes \mathbb{1} \equiv X_2$ or $X \otimes \mathbb{1} \otimes Z \equiv X_1 Z_3$.

(8) Describe the subspace fixed by the stabiliser generators $X \otimes X \otimes \mathbb{1}$ and $\mathbb{1} \otimes X \otimes X$ and its relevance for quantum error correction.

(9) Let S_1 and S_2 be stabiliser generators for a two qubit state $|\psi\rangle$. The state is modified by a unitary operation U. What are the stabiliser generators for $U|\psi\rangle$?

(10) Step through the circuit

Here S is a phase gate: $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto i|1\rangle$.



writing down quantum states of the two qubits after each unitary operation and their respective stabiliser generators. How would you describe the action of the three gates, H, S and controlled-NOT, in the stabiliser language?

Solution:

(1) We will use $|\psi\rangle$ to denote an arbitrary state in the code space.

If $-\mathbb{1}$ is in the stabiliser group, we have:

$$-\mathbb{1}|\psi\rangle = |\psi\rangle$$

which can only be true if $|\psi\rangle = 0$, i.e. we have a trivial code space.

Given any two elements $S, S' \in \mathbb{S}$, since they are both elements of the Pauli group, they either commute or anti-commute. If they anti-commute, we have:

$$|\psi\rangle = SS'|\psi\rangle = -S'S|\psi\rangle = -|\psi\rangle$$

which again can only be true if $|\psi\rangle = 0$, i.e. we have a trivial code space. Hence, to have a non-trivial code space, they must commute.

(2) Trace of tensor product is the product of traces:

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$$

Hence, any Pauli operator that is not the tensor product of identities will have zero trace.

Within the Pauli group, the elements within $\pm i$ phase will square to $-\mathbb{1}$, while the rest will square to $\mathbb{1}$. Since we know that $-\mathbb{1}$ is not in the stabiliser group, the elements in the stabiliser group can only square to $\mathbb{1}$.

(3) Since $S^2 = 1$, we know that all of its eigenvalues are $\lambda_i = \pm 1$. Since any non-identity S is trace zero: $\sum_i \lambda_i = 0$, it must have the same number of $+1$ eigenvectors and -1 eigenvectors.

$\frac{\mathbb{1} \pm S}{2}$ are the projection operators for the $S = \pm 1$ eigen-subspaces.

(4) The effective action of S_2 within the $+1$-eigenspace of S_1 can be represented by:

$$\frac{\mathbb{1} + S_1}{2} S_2 \frac{\mathbb{1} + S_1}{2}$$

Taking the trace we have:

$$\mathrm{Tr}\left(\frac{\mathbb{1} + S_1}{2} S_2 \frac{\mathbb{1} + S_1}{2}\right) = \mathrm{Tr}\left(\frac{\mathbb{1} + S_1}{2} S_2\right) = \frac{1}{2}\left(\mathrm{Tr}(S_2) + \mathrm{Tr}(S_1 S_2)\right) = 0 \quad \text{for } S_1 \neq S_2$$

Hence, within the $+1$-eigenspace of S_1, S_2 must have the same number of $+1$ eigenvectors and -1 eigenvectors. Hence, the $+1$-eigenspace of S_1 is split in half by S_2.

(5) S_1 will split a dim-2^n space into two dim-2^{n-1} eigenspace and we will only focus on one of them, which is the $+1$ eigenspace of S_1. The $+1$ eigenspace of S_1, which is of dim 2^{n-1}, will again split in half by S_2, thus the joint $+1$ eigenspace for both S_1 and S_2 is of dimension 2^{n-2}. Following similar arguments, the joint $+1$ eigenspace for the set of stabiliser generators $\{S_1, S_2, \cdots, S_r\}$ is of dimension 2^{n-r}.

(6) The stabiliser generators for the standard two-qubit basis:

$$\begin{aligned}
|00\rangle &: \quad \{Z_1, Z_2\} \\
|01\rangle &: \quad \{Z_1, -Z_2\} \\
|10\rangle &: \quad \{-Z_1, Z_2\} \\
|11\rangle &: \quad \{-Z_1, -Z_2\}
\end{aligned}$$

The stabiliser generators for the bell states:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle): \quad \{Z_1Z_2, X_1X_2\}$$
$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle): \quad \{-Z_1Z_2, X_1X_2\}$$
$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle): \quad \{-Z_1Z_2, -X_1X_2\}$$
$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle): \quad \{Z_1Z_2, -X_1X_2\}$$

(7) Stabiliser generators: $\{Z_1Z_2, Z_2Z_3\}$. Basis states: $\{|000\rangle, |111\rangle\}$.

(8) It is the subspace spanned by $\{|+++\rangle, |---\rangle\}$, which can be used to correct single-qubit phase-flip (Z) errors.

(9) For a state $|\psi\rangle$ is stabilised by $\{S_i\}$:

$$S_i|\psi\rangle = |\psi\rangle \quad \forall i,$$

then $U|\psi\rangle$ is stabilised by $\{US_iU^\dagger\}$

$$\left(US_iU^\dagger\right)U|\psi\rangle = US_i|\psi\rangle = U|\psi\rangle \quad \forall i,$$

(10)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$|00\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{S_2} \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$$

The starting state $|00\rangle$ is stabilised by the set of stabiliser generators $\{Z_1, Z_2\}$. Following (10), the stabiliser generators will evolve through the circuit in the following way:

$$\begin{aligned}\{Z_1, Z_2\} &\xrightarrow{H_1} \{H_1Z_1H_1, H_1Z_2H_1\} = \{X_1, Z_2\} \\ &\xrightarrow{CNOT} \{\text{CNOT}\cdot X_1\cdot\text{CNOT}, \text{CNOT}\cdot Z_2\cdot\text{CNOT}\} = \{X_1X_2, Z_1Z_2\} \\ &\xrightarrow{S_2} \{S_2X_1X_2S_2^\dagger, S_2Z_1Z_2S_2^\dagger\} = \{X_1Y_2, Z_1Z_2\}\end{aligned}$$

4.7.B **Shor's 9-qubit code.** Use 8 stabiliser generators for Shor's 9-qubit code and explain why this code can correct an arbitrary single-qubit error. In fact, it can also correct some multiple-qubit errors. Which of the following errors can be corrected by the nine-qubit code: X_1X_3, X_2X_7 , X_5Z_6 , Z_5Z_6 , Y_2Z_8?

X_i, Y_i, or Z_i represents X, Y, or Z applied to the i-th qubit.

Solution: Recall that the 9-qubit Shor code and has the set of stabiliser generators:

$$\{Z_1Z_2,\ Z_2Z_3,\ Z_4Z_5,\ Z_5Z_6,\ Z_7Z_8,\ Z_8Z_9,\ X_1X_2X_3X_4X_5X_6,\ X_4X_5X_6X_7X_8X_9\}.$$

Hence, Z_5Z_6 is a stabiliser that would act trivially on our code state.

The 9-qubit Shor code is just putting a bit-flip code within a phase-flip code. Hence, the block of qubit 1,2,3 would form a bit flip code and any single-qubit bit-flip occur within block of qubit 1,2,3 can be corrected. Similarly for the block of qubit 4,5,6 and for the block of qubit 7,8,9.

X_1X_3 cannot be corrected since these are *two* bit-flip errors within the block of qubit 1,2,3.

X_2X_7 can be corrected since these are *one* bit-flip error within the block of qubit 1,2,3 and one bit-flip within the block of qubit 7,8,9.

Now since the phase-flip code is the upper layer and span all of 9 qubits, any single-qubit phase-flip on any of the 9 qubits can be corrected.

X_5Z_6 is a single-qubit bit-flip error that can be corrected by the block of qubit 4,5,6, and a single-qubit phase-flip error that can be corrected by the phase flip code.

$Y_2Z_8 \propto X_2 \cdot Z_2Z_8$, which means two phase-flip errors within the code, which cannot be corrected.

Fun question: does our conclusion change if we put phase flip code under the bit-flip code instead?